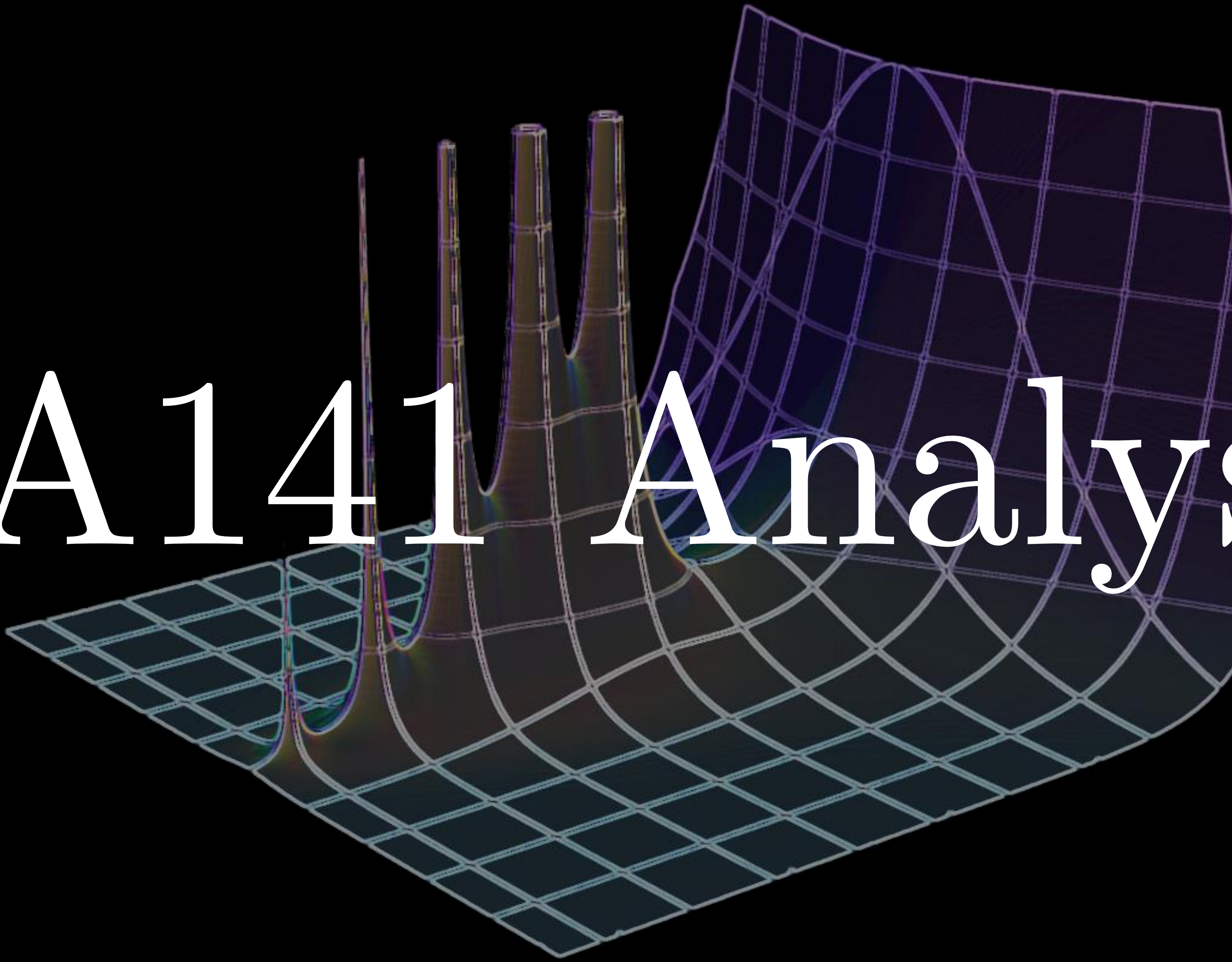
MA141 Analysis I



Continuity

A function $f : X \to \mathbb{R}$ is *continuous* at c if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Exercise. Prove that sin is continuous.

Proof. Let $\varepsilon > 0$. Then, if $|x-c| < \delta$, we have:

$$|f(x) - f(c)| = |\sin(x) - \sin(c)|$$

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By the addition formulae

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So, if $\delta = \varepsilon$, we have:

$$= \varepsilon$$

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&= |x - c| \\
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So, if $\delta = \varepsilon$, we have:

$$= \varepsilon$$

Exercise. Prove that sin is continuous.

Proof. Let $\varepsilon > 0$, and pick $\delta = \varepsilon$. Then, if $|x - c| < \delta$, we have:

$$\begin{aligned}
|f(x) - f(c)| &= |\sin(x) - \sin(c)| \\
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&\leq \left|2\left(\frac{x-c}{2}\right)\right| && |\sin(x)| \leq |x| \\
&= |x - c| \\
&< \delta \\
&= \varepsilon
\end{aligned}$$

Exercise. Prove that the Heaviside step function is discontinuous at 0.

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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A function $f : X \to \mathbb{R}$ is *discontinuous* at c if

there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in X$ such that

$$|x - c| < \delta \quad \text{but} \quad |f(x) - f(c)| \not< \varepsilon$$

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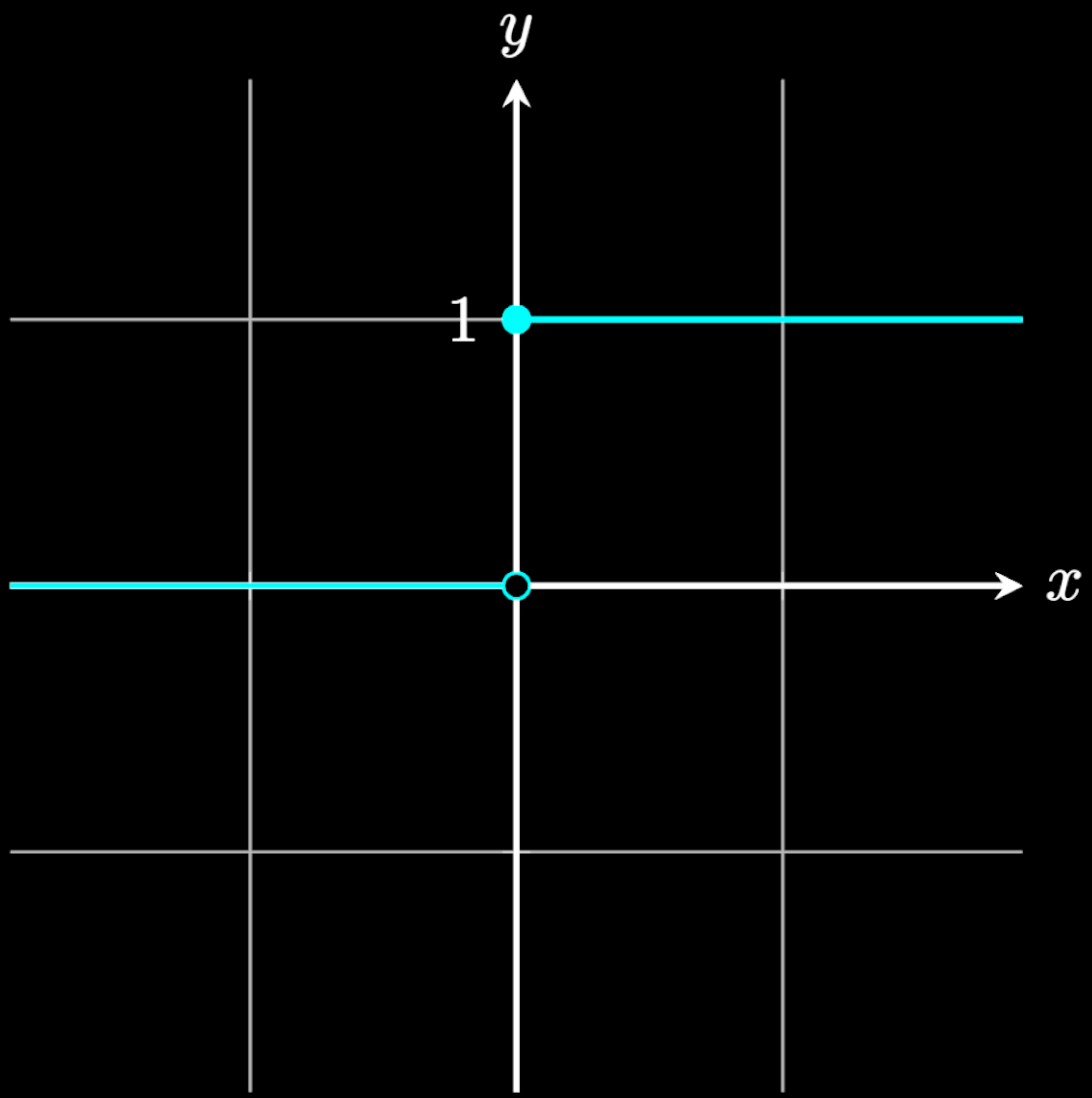

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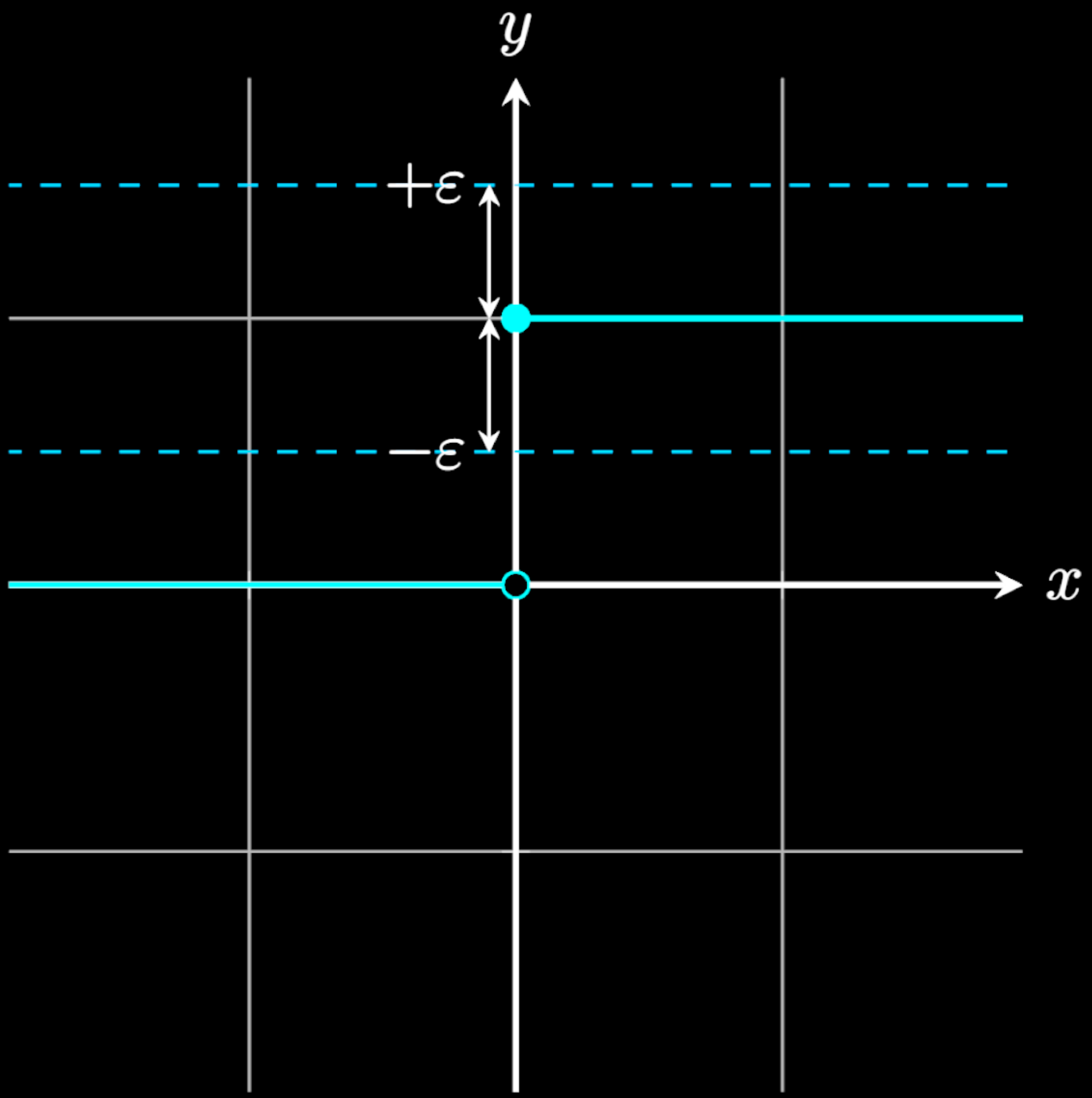

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Proof. Let $\varepsilon = \frac{1}{2}$, $\delta > 0$ and choose $x = -\frac{\delta}{2}$. Then,

$$\begin{aligned} |x - c| &= \left| -\frac{\delta}{2} - 0 \right| \\ &= \frac{\delta}{2} \\ &< \delta \end{aligned}$$

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$$\begin{aligned} |x - c| &= \left| -\frac{\delta}{2} - 0 \right| \\ &= \frac{\delta}{2} \\ &< \delta \end{aligned}$$

But,

$$\begin{aligned} |f(x) - f(c)| &= |0 - 1| \\ &= 1 \\ &\not< \varepsilon \end{aligned}$$

so f is discontinuous at 0.

Continuity

A function $f : X \to \mathbb{R}$ is *continuous* at c if

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Continuity

A function $f : X \to \mathbb{R}$ is *sequentially continuous* at c if

for all sequences $(x_n) \subseteq X$ that converge to c,

$$\lim_{n\to\infty} f(x_n) = f(c)$$

Exercise. Determine at which point(s) the following function is continuous:

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

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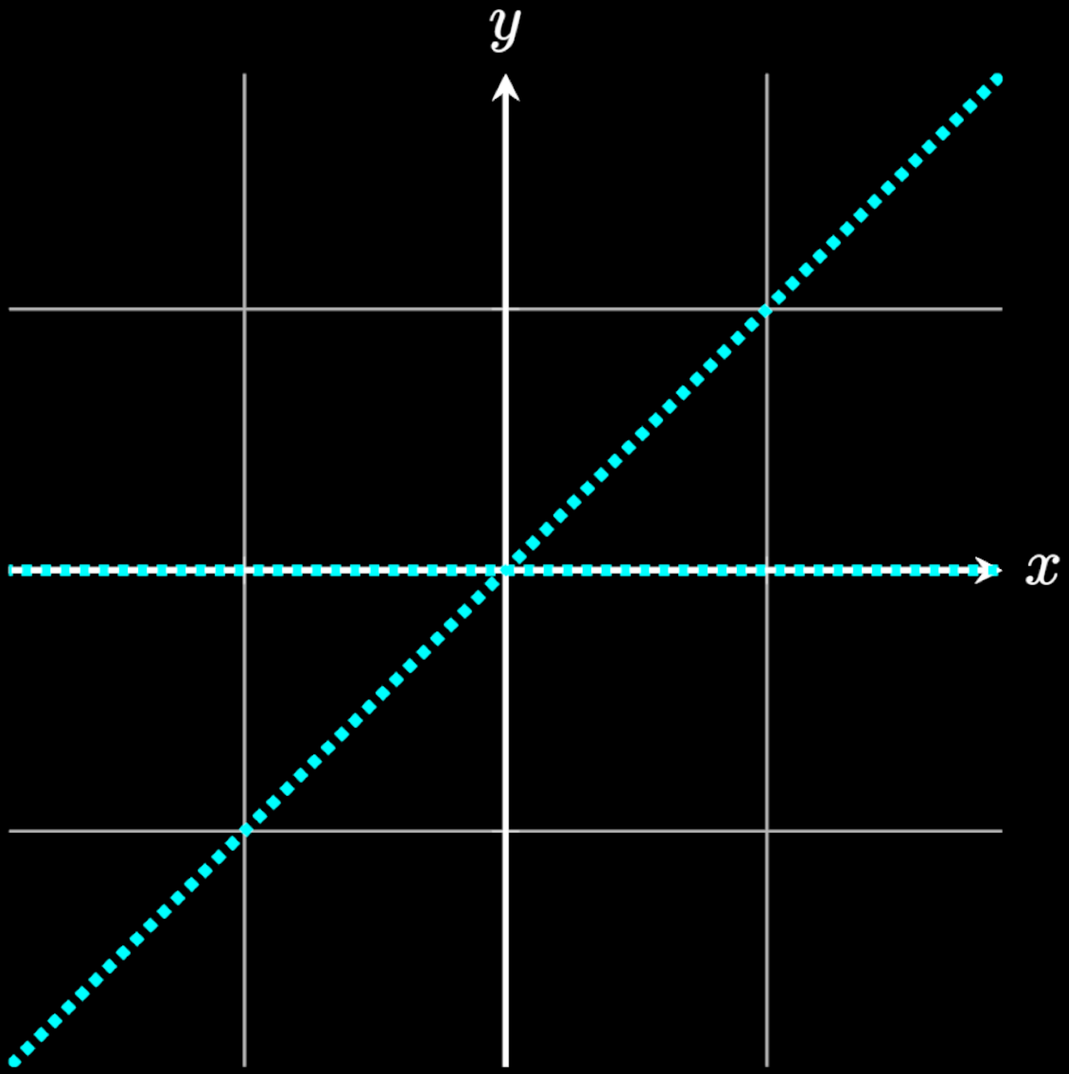


Proof. Let $c = 0$ and let (x_n) be a sequence converging to c. Then,

$$\begin{aligned} \lim_{n \to \infty} f(x_n) &= 0 \\ &= f(0) \end{aligned}$$

so f is continuous at 0.

Proof. Let $c = 0$ and let (x_n) be a sequence converging to c. Then,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= 0 \\ &= f(0)\end{aligned}$$

so f is continuous at 0. Suppose otherwise that $c \in \mathbb{Q}^*$, and let $(x_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ be defined by:

$$x_n := c + \frac{\sqrt{2}}{n}$$

Then,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} 0 \\ &= 0 \\ &\neq c \\ &= f(c)\end{aligned}$$

so f is discontinuous on $\mathbb{Q}^*$.

Now suppose $c \in \mathbb{R} \setminus \mathbb{Q}$ and let $(x_n) \subseteq \mathbb{Q}$ be defined by:

$$x_n := \frac{\lfloor c \cdot 10^n \rfloor}{10^n}$$

Then,

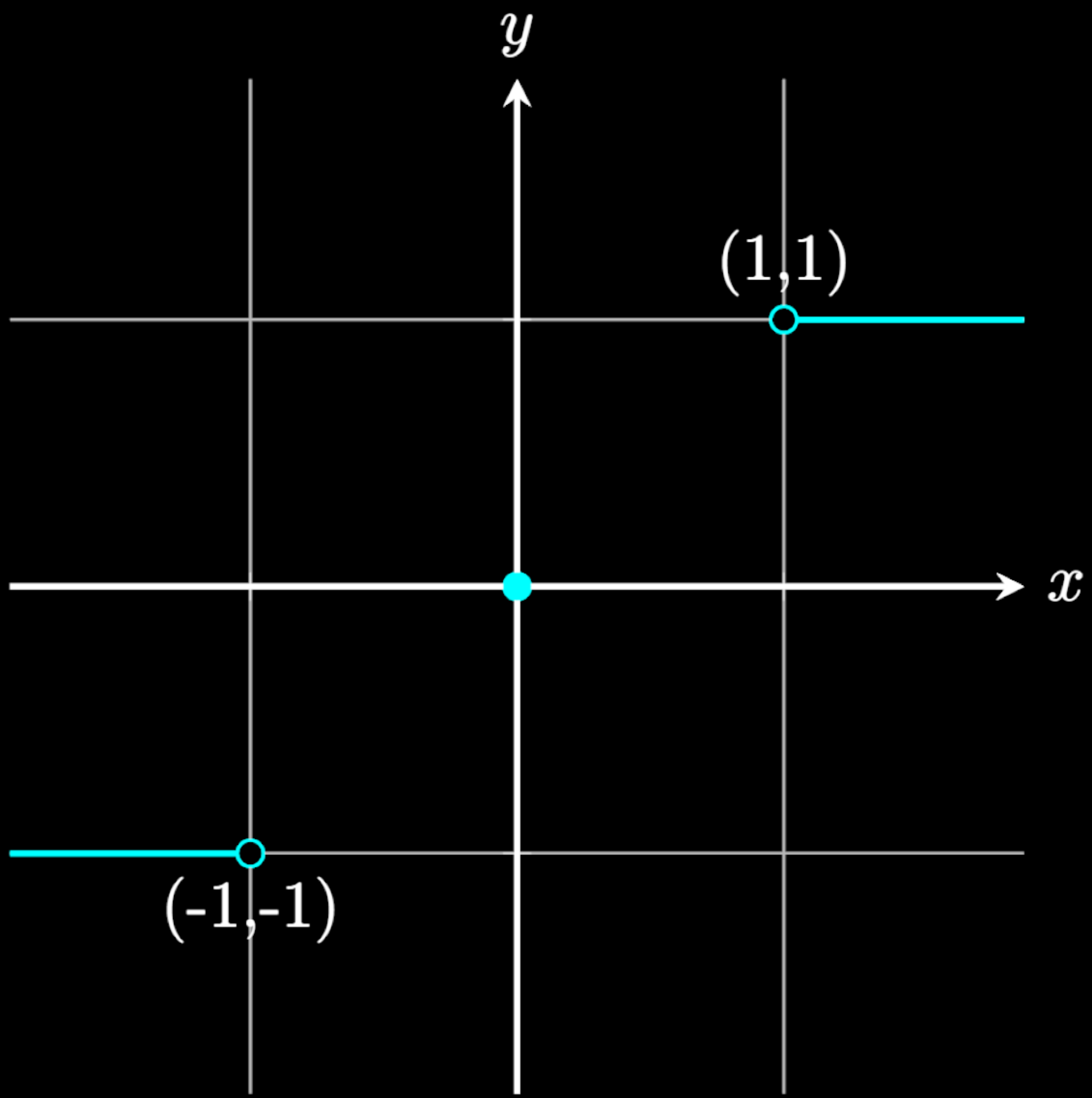
$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n \\ &= c \\ &\neq 0 \\ &= f(c)\end{aligned}$$

so f is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Exercise. Determine at which point(s) the following function is continuous:

$$f : (-\infty, -1) \cup \{0\} \cup (1, \infty) \to \mathbb{R}$$

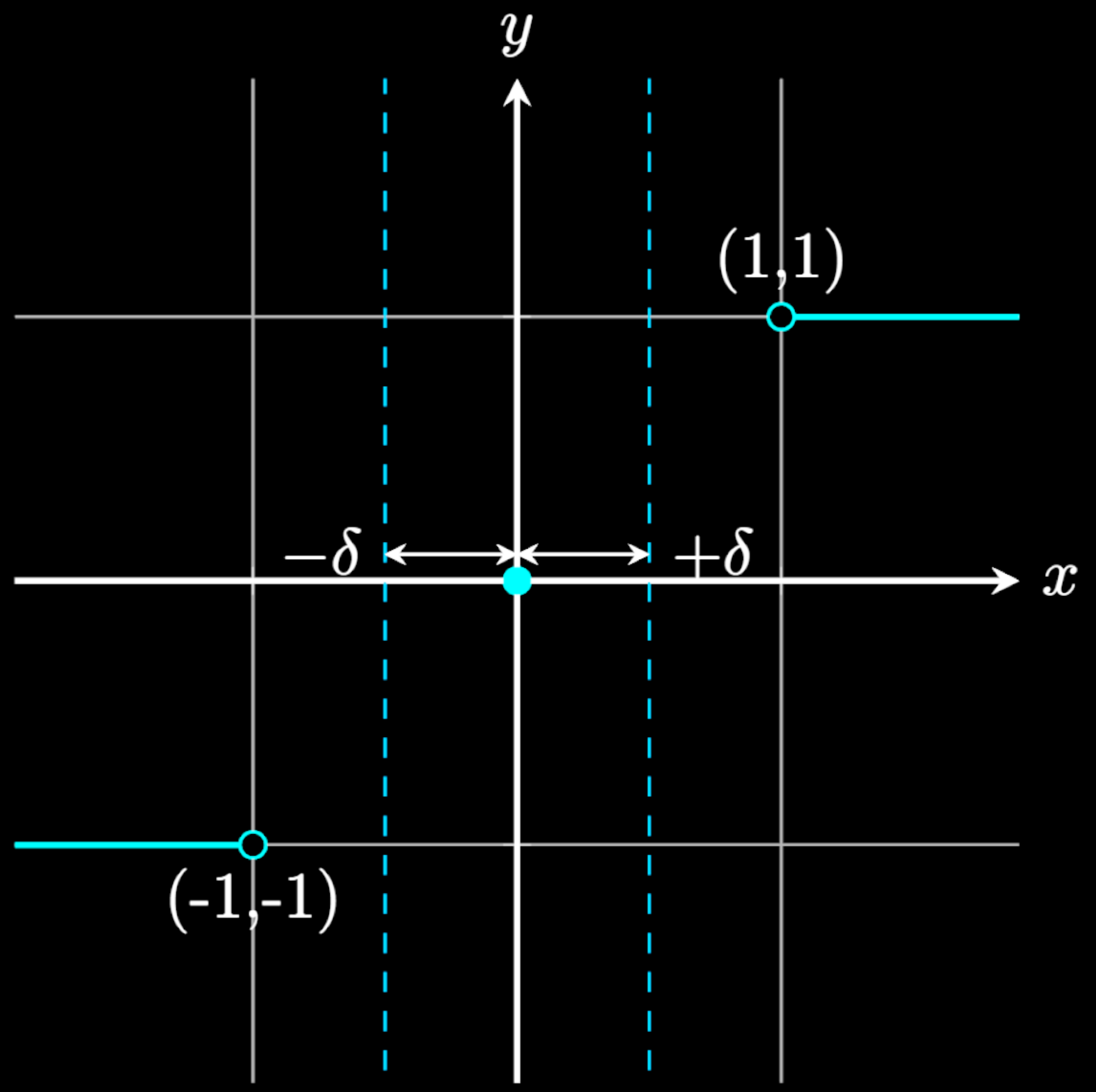
$$f(x) = \begin{cases} -1 & x < -1 \\ 0 & x = 0 \\ 1 & x > 1 \end{cases}$$



Exercise. Determine at which point(s) the following function is continuous:

Consider $c = 0$ and let $\varepsilon > 0$.

Pick $\delta = \frac{1}{2}$.

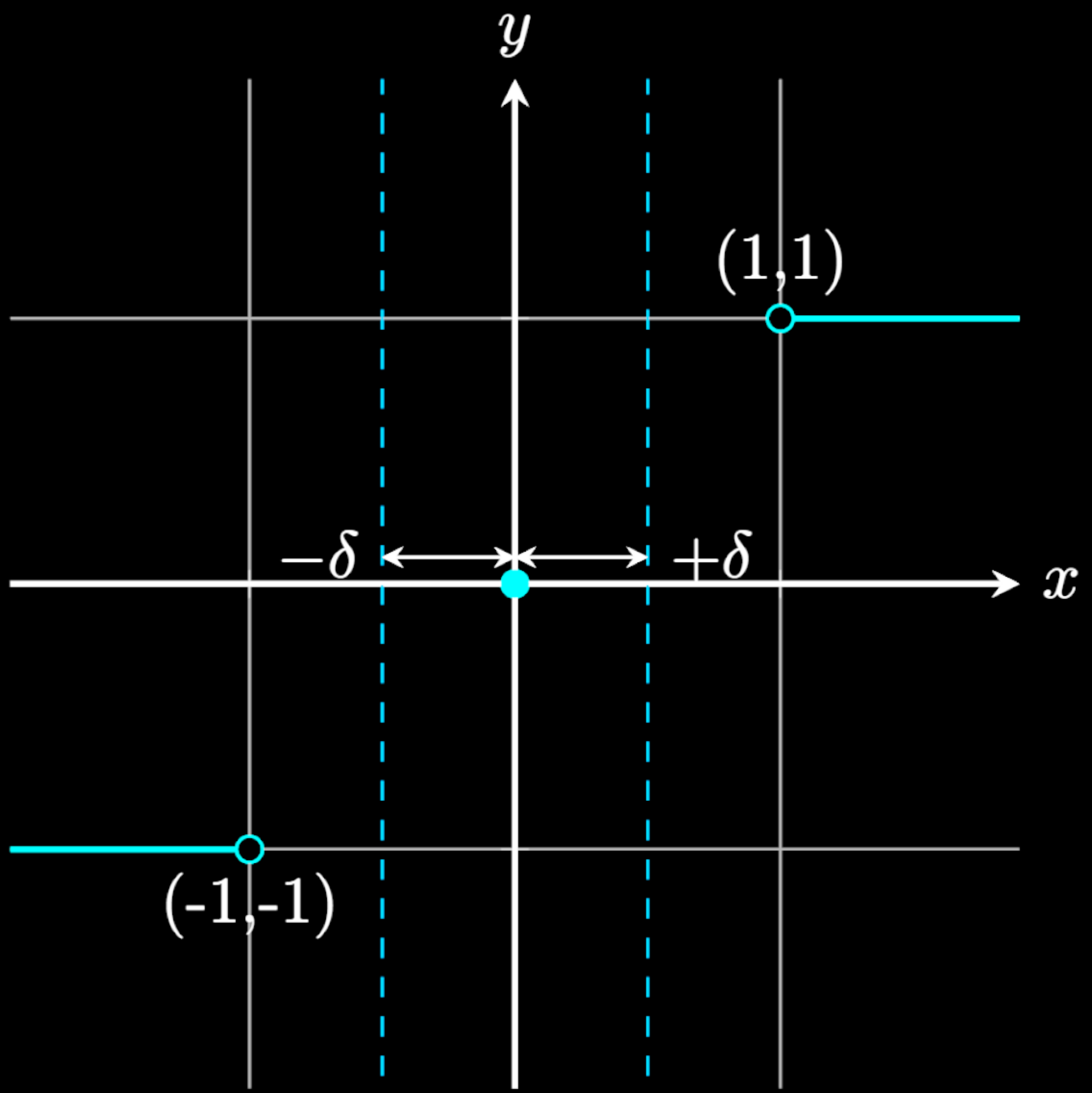


Exercise. Determine at which point(s) the following function is continuous:

Consider $c = 0$ and let $\varepsilon > 0$.
Pick $\delta = \frac{1}{2}$. Then, if $|x - c| < \delta$, we necessarily have $x = c$, and hence:

$$\begin{aligned} |f(x) - f(c)| &= |f(c) - f(c)| \\ &= 0 \\ &< \varepsilon \end{aligned}$$

so f is continuous at 0.



An aside from Aris

– intermission –

Recall from Assignment 2, Q12:

Define a sequence (a_n) by:

$$a_1 = \sqrt{3}$$
$$a_{n+1} = \sqrt{a_n + 2}$$

Assume that $(a_n) \to \ell$, and deduce the value of ℓ.

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 2}$$
$$\ell = \sqrt{\ell + 2}$$

Define a sequence (a_n) by:

$$a_1 = 1$$
$$a_{n+1} = \operatorname{sgn}\left(\frac{a_n}{n}\right)$$

Assume that $(a_n) \to \ell$, and deduce the value of ℓ.

$$\lim_{n\to\infty} \operatorname{sgn}\left(\frac{a_n}{n}\right) = \lim_{n\to\infty} 1$$
$$= 1$$

$$\operatorname{sgn}\left(\lim_{n\to\infty} \frac{a_n}{n}\right) = \operatorname{sgn}(0)$$
$$= 0$$

End of the aside

– intermission end –

Intermediate Value Theorem

Theorem (IVT). Suppose that f is continuous on $[a, b]$, and that $f(a) < f(b)$. For any k satisfying $f(a) < k < f(b)$ there exists $c \in (a, b)$ such that $f(c) = k$.

Exercise. Let $f: [1, 3] \to \mathbb{R}$ be continuous, satisfying $f(1) = 2, f(2) = 3,$ *and* $f(3) = 1.$ Prove that f has a fixed point in $[1, 3]$.

Exercise. Let $f:[1,3] \to \mathbb{R}$ be continuous, satisfying $f(1) = 2$, $f(2) = 3$, *and* $f(3) = 1$. Prove that f has a fixed point in $[1,3]$.

Proof. Define $g:[1,3] \to \mathbb{R}$ by

$$g(x) := f(x) - x$$

Note that g is continuous as the sum of continuous functions, and that $g(3) < 0$ and $g(1) > 0$. By the IVT, there exists $c \in (1,3)$ such that $g(c) = 0$, so $f(c) = c$.

Extreme Value Theorem

Theorem (EVT). Suppose that f is continuous on $[a, b]$. Then, f is bounded and attains its bounds. That is, there exist numbers $x_*, x^* \in [a, b]$ such that for all $x \in [a, b]$, we have $f(x_*) \leq f(x) \leq f(x^*)$.

Exercise. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous, and there exist $M, R > 0$ such that for all $x \geq R$, f satisfies $|f(x)| \leq M$. Prove that f is bounded.

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Proof. Consider the restriction of f to $[0, R]$. By the EVT, this restriction is bounded. That is, there exists $M' > 0$ such that for all $x \in [0, R]$, we have

$$|f(x)| \leq M'$$

Then, $M'' = \max\{M, M'\}$ bounds f everywhere.

Completeness Axioms

Axiom (LUB). Every bounded above increasing sequence converges.

Axiom (GLB). Every bounded below decreasing sequence converges.

Axiom (Cauchy Completeness). Every Cauchy sequence converges.

Axiom (B–W). (Covered later.)

Exercise. Every bounded below decreasing sequence converges to its infimum.

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Proof. Let (a_n) be a bounded below decreasing sequence, and let L be its infimum. Let $\varepsilon > 0$, and consider $L + \varepsilon$.

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Proof. Let (a_n) be a bounded below decreasing sequence, and let L be its infimum. Let $\varepsilon > 0$, and consider $L + \varepsilon$.

Because L is the infimum, there exists N such that $L \leq a_N < L + \varepsilon$. Because the sequence is decreasing, all following terms also satisfy this inequality, so we have

$$|a_n - L| < \varepsilon$$

for all $n > N$.

Bolzano–Weierstrass Theorem

Theorem (B–W). Any bounded sequence of real numbers has a convergent subsequence.

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Proof. If (a_n) does not diverge to infinity, then there exists $M > 0$ such that for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $a_n \leq M$. We now construct a subsequence by picking one such term of (a_n) for each $N \in \mathbb{N}$.

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By the Bolzano–Weierstrass theorem, this subsequence has a convergent subsequence, say $\left(q_{n_{k_i}}\right)$. Suppose this converges to some $q \in \mathbb{N}$.

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Consider the corresponding subsequence $(p_{n_{k_i}})$. We wish to show that this converges to αq. Note that $(p_{n_{k_i}})$ must also converge to a natural number.

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Consider the corresponding subsequence $(p_{n_{k_i}})$. We wish to show that this converges to αq. Note that $\left(p_{n_{k_i}}\right)$ must also converge to a natural number.

We can write $p_{n_{k_i}} = \frac{p_{n_{k_i}}}{q_{n_{k_i}}} q_{n_{k_i}}$ and then use the product rule for convergent sequences.

But α is irrational, so αq is irrational.

Series Convergence Tests

$$\sum_{n=1}^{\infty} \frac{n^2+3n+1}{(n^2+3)^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$$

$$\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$$